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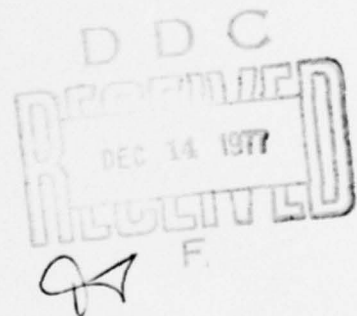
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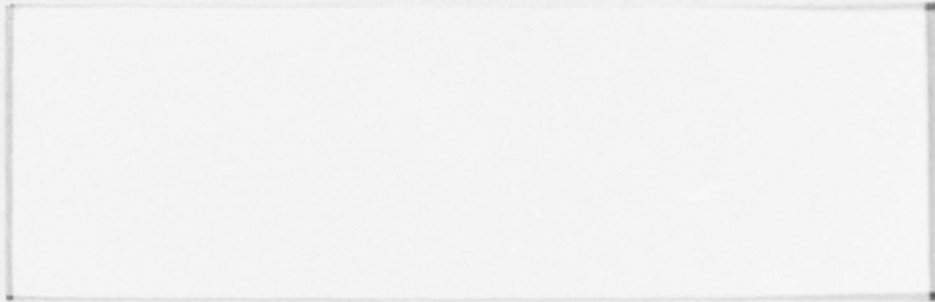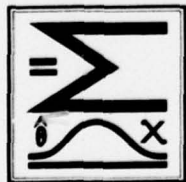
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Multistate Coherent Systems

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ABSTRACT

Multistate Coherent Systems.

The vast majority of reliability analyses assume that components and
system are in either of two states: functioning or failed. The present paper
develops basic theory for the study of systems of components in which any of
a finite number of states may occur, representing at one extreme perfect
functioning and at the other extreme complete failure. We lay down axioms
extending the standard notion of a coherent system to the new notion of a
multistate coherent system. For such systems we obtain deterministic and
probabilistic properties for system performance which are analogous to well-
known results for coherent system reliability.

Multistate Coherent Systems.

1. Introduction and Summary

A central problem in reliability theory is to determine the relationship
between the reliability of a complex system and the reliabilities of its
components. Thus far, in practically all treatments of this problem, the
system and its components are considered to be in either of two states:
functioning (denoted by 1) and failed (denoted by 0). The theory of binary
coherent structures has served as a unifying foundation for a mathematical
and statistical theory of reliability for this dichotomous case. In fact,
fairly complete solutions of various aspects of this problem have been
obtained by Birnbaum, Esary, Saunders, Marshall, Barlow, and Proschan. See,
for example, [1], [3], [4], [5], [6], and [7].

In many real life situations, however, the systems and their components
are capable of assuming a whole range of levels of performance, varying from
perfect functioning (denoted by level M, say) to complete failure (denoted by
0). In these situations, the dichotomous model is an oversimplification of
the actual situation, and so models representing multistate systems and
multistate components are much more useful in describing the performance of
these systems in terms of the performance of their components.

Unfortunately, very little work has been done on this more general
problem of multistate systems. Among the earlier papers treating aspects of
multistate systems are [8], [9], [10], [11], [13], and [14]. With the
exception of [10] and [11], these papers deal mainly with models for
cannibaliz   n, and barely touch on the performance of systems and components
assuming   two states. More recent and more sophisticated work on

multistate systems has been performed by Barlow [2] and Ross [12]; however, their models are more specialized than ours, as we shall see.

The main purpose of the present paper is to develop an adequate general model and theory for the case in which both systems and components may assume any of an ordered set of states, say $0,1,2,\ldots,M$; this theory generalizes coherent structure theory. We develop the concept of a multistate coherent structure as a generalization of the well-known concept of binary coherent structure. We then use this concept as a unifying foundation for the study of the relationship between the performance of a sytem and the performance of the components in the system. In forthcoming papers, we shall present treatments of various stochastic aspects of multistate systems with the aim of ultimately achieving a comprehensive theory analogous to coherent structure theory in the binary case (see [1]).

We now summarize the contents of this paper. Our formulation and treatment are similar to that of Barlow and Proschan [1] for the two state case. In Section 2 we present the notation and terminology used throughout the paper. In Section 3 we consider a system of n components. For each component and for the system itself, we can distinguish among say $M+1$ states representing successive levels of performance ranging from perfect functioning (level M) down to complete failure (level 0). For component i, x_i denotes the corresponding state or performance level, $i=1,2,\ldots,n$; the vector $\underline{x} = (x_1,\ldots,x_n)$ denotes the vector of states of components $1,\ldots,n$. We assume that the state ϕ of the system is a deterministic function of the states $x_1,\ldots,x_n$ of the components. Thus $\phi = \phi(\underline{x})$, where $\underline{x}$ assumes its values in S^n, $S = \{0,1,\ldots,M\}$, and ϕ assumes its values in S. We define a multistate coherent sturcture as a natural generalization of the standard

concept of a binary coherent structure by requiring three reasonable conditions that ϕ must satisfy.

We then obtain deterministic relationships between the performance of a system and the performance of its components; these relationships are natural generalizations of well-known results in the binary case. Thus we show that the performance of a multistate coherent system is bounded below by the performance of a series system and bounded above by the performance of a parallel system. We next present a decomposition identity useful in deriving inductive proofs and probabilistic properties for systems. Finally, we generalize the practical result that redundancy at the component level is better than redundancy at the system level.

In Section 4 we investigate the probabilisitic aspects of multistate coherent systems. We relate in a probabilistic sense the performance of the system to the performance of its components, assumed statistically independent. Next the decomposition identity of Section 3 is used to obtain a corresponding decomposition identity for the performance function of the system. This decomposition identity is then used to show that system performance is a monotone increasing function of component performances. We end the section by obtaining bounds on system performance.

Finally, in Section 5 we study some dynamic aspects of multistate coherent systems. In earlier sections, we tacitly assume that time is fixed. In Section 5 we consider multistate coherent system as operating over time. At time 0 the system and each of its components are in state M (corresponding to perfect functioning). As time passes, the performance level of components (and consequently of the system) deteriorates to lower levels until finally level 0 (complete failure) is reached. We define the

concepts of IFRA and NBU stochastic processes introduced by Ross [12]. We present a different definition for an NBU stochastic process, and prove the analogue of the NBU closure theorems using a new characterization of the NBU property.

2. Notation and Terminology.

The vector $\underline{x} \equiv (x_1, \ldots, x_n)$ denotes the vector of states of components $1, \ldots, n$.

$S = \{0, 1, \ldots, M\}$ denotes the set of possible states of both components and systems

$C = \{1, 2, \ldots, n\}$ denotes the set of component indices.

$(j_i, \underline{x}) \equiv (x_1, \ldots, x_{i-1}, j, x_{i+1}, \ldots, x_n)$, where $j = 0, 1, \ldots, M$.

$(\cdot_i, \underline{x}) \equiv (x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)$.

$\underline{j} \equiv (j, j, \ldots, j)$.

$\underline{y} < \underline{x}$ means $y_i \leq x_i$ for $i = 1, \ldots, n$ and $y_i < x_i$ for some i.

$x \vee y$ denotes $\max(x, y)$.

$\underline{x} \vee \underline{y} \equiv (x_1 \vee y_1, \ldots, x_n \vee y_n)$.

$x \wedge y$ denotes $\min(x, y)$.

$\underline{x} \wedge \underline{y} \equiv (x_1 \wedge y_1, \ldots, x_n \wedge y_n)$.

"increasing" is used in place of "nondecreasing", and "decreasing" is used in place of "nonincreasing".

When we say $f(x_1, \ldots, x_n)$ is increasing we mean f is increasing in each argument.

Given a univariate distribution F, its complement 1-F is denoted by $\bar{F}$.

3. Deterministic Properties of Multistate Coherent Systems.

Consider a system of n components. We assume that for both the system and its components we can distinguish among a finite number of distinct

states representing various levels of performance, ranging from perfect functioning (state M) to complete failure (state 0). As time passes a component, starting in state M, deteriorates and enters state M-1, deteriorates further entering state M-2, etc., until ultimately it descends to state 0; a similar succession of decreasing state levels describes the system progression over time. (In a later paper, we consider the case of a continuous range of performance levels varying over the interval $[0,1]$.)

The performance x_i of component i assumes a value in the set $S = \{0,1,\ldots,M\}$. We assume that the performance of the systems depends deterministically on the performance of each of the components. Thus the state of the system is determined by a function $\phi: S^n \to S$. Given $\underline{x}$, the vector of component states, we may determine $\phi(\underline{x})$, the system state. The function ϕ is called the structure function of the system.

The structure function ϕ satisfies certain conditions that represent intuitively reasonable properties of systems encountered in practice. In the binary case the following two conditions are required for a system to be a coherent structure ([1], Def. 2.1, p. 6):

    (i) The function $\phi(\underline{x})$ is increasing.

    (ii) Each component is relevant to the system; i.e., for each i there exists a vector $(\cdot_i,\underline{x})$ such that $\phi(1_i,\underline{x}) > \phi(0_i,\underline{x})$. This means that the function ϕ is not constant in the ith argument, $i=1,\ldots,n$.

Condition (i) embodies the reasonable assumption that improving the performance of a component is not harmful to system performance. Condition (ii) eliminates from consideration components which have no effect on system performance.

In our present multistate model, we stipulate three conditions:

Definition 3.1. A system of n components is said to be a _multistate coherent system_ (MCS) if its structure function ϕ satisfies:

(i)'. ϕ is increasing.

(ii)'. For level j of component i, there exists a vector $(\cdot_i, \underline{x})$ such that $\phi(j_i, \underline{x}) = j$ while $\phi(\ell_i, \underline{x}) \neq j$ for $\ell \neq j$, i=1,...,n and j=0,1,...,M.

(iii)'. $\phi(\underline{j}) = j$ for j=0,1,...,M.

Note that conditions (i)' and (ii)' generalize conditions (i) and (ii) in the binary case. Condition (iii)' is automatically satisfied in the binary case, but is not implied in the present multistate case by (i)' and (ii)'.

Some examples of MCS's are:

Example 3.1. A series system: $\phi(\underline{x}) = \min_{1 \leq i \leq n} x_i$.

Example 3.2. A parallel system: $\phi(\underline{x}) = \max_{1 \leq i \leq n} x_i$.

Example 3.3. A k-out-of-n structure: $\phi(\underline{x}) = x_{(n-k+1)}$, where $x_{(1)} \leq \cdots \leq x_{(n)}$ is an increasing rearrangement of $x_1,...,x_n$.

Example 3.4. Let $P_1,...,P_r$ be nonempty subsets of $C = \{1,...,n\}$ such that $\bigcup_{i=1}^{r} P_i = C$ and $P_i \not\supseteq P_j$, i≠j. Let $\phi(\underline{x}) = \max_{1 \leq j \leq r} \min_{i \in P_j} x_i$. Then $\phi(\underline{x})$ is a MCS, and $P_1,...,P_r$ are called its min path sets.

Remark 3.1. The structures in Examples 3.1, 3.2, and 3.3 are natural generalizations of familiar basic structures in the binary case. They constitute special cases of the structure in Example 3.4, which in the binary case defines the most general binary coherent structure ([1], Chap. 1). The structure function of Example 3.4 is due to Barlow [2]. Since the structure functions of Example 3.4 satisfy conditions (i)', (ii)', and (iii)' of

Definition 3.1, they constitute a subclass of our MCS class. By examining

some special cases it is easy to see that the class in Example 3.4 is actually

a small subclass of our MCS. For instance, for a two component system,

Example 3.4 yields only two systems: the parallel system and the series

system. However for $S = \{0,1,2\}$ there are more than 12 MCS's.

In the remainder of this section we investigate the structural properties

of the MCS. We extend results obtained in the binary case ([1], Chap. 1) to

the more general multistate case.

The following lemma gives a decomposition identity useful in carrying

out inductive proofs. It holds for any multistate structure, not just for

the MCS.

Lemma 3.1. The following identity holds for any n-component structure

function ϕ.

(3.1)
$$\phi(\underline{x}) = \sum_{j=0}^{M} \phi(j_i, \underline{x}) \, I_{[x_i=j]} \quad \text{for } i=1,\ldots,n,$$

where

$$I_{[x_i=j]} = \begin{cases} 1 \text{ if } x_i=j \\ 0 \text{ if } x_i \neq j. \end{cases}$$

The proof is obvious and therefore omitted.

The following theorem gives simple bounds on MCS performance.

Theorem 3.1. Let ϕ be the structure function of an MCS of n components.

Then

(3.2)
$$\min_{1 \leq i \leq n} x_i \leq \phi(\underline{x}) \leq \max_{1 \leq i \leq n} x_i.$$

Proof. Let $m = \max_{1 \leq i \leq n} x_i$. Then $\phi(\underline{x}) \leq \phi(\underline{m})$ by the monotonicity of ϕ.

By condition (iii)' of Definition 3.1, $\phi(\underline{m}) = m$. The upper bound follows.

The proof establishing the lower bound is similar. $\|$

Theorem 3.1 states that a parallel system yields the best performance of an MCS, and a series system yields the worst performance. Using this theorem, we will show similar probabilistic bounds in Section 4.

As in the binary case, we may define a dual structure for each multistate structure.

Definition 3.2. Let ϕ be the structure function of a multistate system. The dual structure function ϕ^D is given by:

$$(3.3) \qquad \phi^D(\underline{x}) = M - \phi(M-x_1,\ldots,M-x_n).$$

It is easy to verify that the dual of an MCS is an MCS.

Example 3.5. The dual of a series (parallel) system is a parallel (series) system. More generally, the dual of a k-out-of-n system is an (n-k+1)-out-of-n system.

Design engineers have used the well-known principle that redundancy at the component level is preferable to redundancy at the system level (all other things being equal). We present this principle in mathematical form along with a proof for MCS's.

Theorem 3.2. Let ϕ be the structure function of an MCS. Then

(i). $\phi(\underline{x}V\underline{y}) \geq \phi(\underline{x}) \ V \ \phi(\underline{y})$.

(ii). $\phi(\underline{x}\wedge\underline{y}) \leq \phi(\underline{x}) \ \wedge \ \phi(\underline{y})$.

Equality holds in (i) for all $\underline{x}$ and $\underline{y}$ if and only if the structure is parallel. Equality holds in (ii) for all $\underline{x}$ and $\underline{y}$ if and only if the structure is series.

Proof. (i). $x_i \ V \ y_i \geq x_i$, i=1,...,n. Thus $\phi(\underline{x}V\underline{y}) \geq \phi(\underline{x})$ since ϕ is increasing. Similarly, $\phi(\underline{x}V\underline{y}) \geq \phi(\underline{y})$. It follows that $\phi(\underline{x}V\underline{y}) \geq \max[\phi(\underline{x}), \phi(\underline{y})] \equiv \phi(\underline{x}) \ V \ \phi(\underline{y})$.

(ii). A similar argument proves (ii).

If the structure is parallel (series), then equality in (i) ((ii)) is readily established.

Next assume $\phi(\underline{x}V\underline{y}) = \phi(\underline{x}) \ V \ \phi(\underline{y})$ for all $\underline{x}$ and $\underline{y}$. For each i,j, there exists $(\cdot_i, \underline{x})$ such that $\phi(j_i, \underline{x}) = j$ and $\phi(\ell_i, \underline{x}) < j$ when $\ell < j$, i=1,...,n and j = 0,1,...,M (by (ii)' of Def. 3.1). Since $(j_i, \underline{x}) = (j_i, 0) \ V \ (0_i, \underline{x})$, we have $j = \phi(j_i, \underline{x}) = \phi(j_i, \underline{0}) \ V \ \phi(0_i, \underline{x})$. It follows that $\phi(j_i, \underline{0}) = j$ for i=1,...,n and j=0,1,...,M. Now $\phi(\underline{x}) = \phi(x_1, 0, ..., 0) \ V \ \phi(0, x_2, 0, ..., 0) \ V \ ... \ V \ \phi(0, 0, ..., x_n) = x_1 V x_2 V ... V x_n \equiv \max_{1 \le i \le n} x_i$. Therefore ϕ is a parallel structure.

Finally, to prove necessity for equality in (ii), assume $\phi(\underline{x}\wedge\underline{y}) = \phi(\underline{x}) \ \wedge \ \phi(\underline{y})$ for all $\underline{x}$ and $\underline{y}$. Let ϕ^D be the dual MCS of ϕ. It is easy to show that $\phi^D(\underline{x}V\underline{y}) = \phi^D(\underline{x}) \ V \ \phi^D(\underline{y})$ for all $\underline{x}$ and $\underline{y}$. Hence by (i), ϕ^D is a parallel structure. Therefore ϕ is a series structure. $\|$

In binary coherent structure theory, the concepts of minimal path vectors and minimal cut vectors play a crucial role. The analogue in MCS theory is the concept of critical connection vectors. Using this concept we can represent the state of a MCS in terms of its critical connection vectors.

Definition 3.3. A vector $\underline{x}$ is said to be a connection vector to level j if $\phi(\underline{x}) = j$, j=0,1,...,M.

Definition 3.4. A vector $\underline{x}$ is said to be an upper critical connection vector to level j if $\phi(\underline{x}) = j$ and $\underline{y} < \underline{x}$ implies $\phi(\underline{y}) < j$, j=1,...,M.

Similarly, we can define a lower connection vector to level j, j=0,1,..., M-1.

The existence of such critical connection vectors is guaranteed by the conditions of Definition 3.1.

Let $\underline{x}$ be an upper critical connection vector to level j. Define $C_j(\underline{x})$ = {i: $x_i \geq j$}. Obviously $C_j(\underline{x})$ is a non-empty subset of C = {1,...,n}. For j=1,...,M, let $\mathcal{C}_j$ = {$C_j(\underline{x})$: $\underline{x}$ is an upper critical connection vector to level j}. Then the following lemma shows that $\mathcal{C}_j$ enjoys a property similar to that enjoyed by the minimal path sets and the minimal cut sets in the binary case.

Lemma 3.2. For j=1,...,M,

    (i). $U\mathcal{C}_j$ = {1,2,...,n}.

    (ii). If A and B are two different members of $\mathcal{C}_j$, then A⊅B.

The proof follows readily from Def. 2.1 and hence is omitted.

For j=1,...,M, let $\underline{y}_1^j,...,\underline{y}_{n_j}^j$ be the upper critical connection vectors to level j, where $\underline{y}_r^j = (y_{1r}^j,...,y_{nr}^j)$. The following theorem, stated without proof, enables us to determine the state of an MCS using its upper critical connection vectors.

Theorem 3.3. Let ϕ be the structure function of an MCS. Let $\underline{y}_1^j,...,\underline{y}_{n_j}^j$ be its upper critical connection vectors to level j, j=1,...,M. Then (3.4) $\phi(\underline{x}) \geq j$ if and only if $\underline{x} \geq \underline{y}_\ell^j$ for some ℓ, $1 \leq \ell \leq n_j$.

Theorem 3.3 is used in Section 4 to establish bounds on the system performance function.

4. Stochastic Performance of Multistate Coherent Systems.

In Section 3 we discussed deterministic aspects of MCS's. In this section, we determine the relationship between the stochastic performance of the system and the stochastic performances of its components. We also obtain bounds on system performance which are particularly useful when exact system performance is difficult to evaluate.

Let X_i denote the random state of componenet i, with

(4.1)
$$P[X_i = j] = p_{ij} ,$$
$$P[X_i \leq j] = P_i(j) ,$$

$j = 0, 1, \ldots, M$ and $i = 1, \ldots, n$. P_i represents the performance distribution of component i. Clearly

$$P_i(j) = \sum_{k=0}^{j} p_{ik} ,$$

$$P_i(M) = \sum_{k=0}^{M} p_{ik} = 1 ,$$

for $i = 1, \ldots, n$.

Let $\underline{X} = (X_1, \ldots, X_n)$ be the random vector representing the states of components $1, \ldots, n$, where the $X_1, \ldots, X_n$ are assumed to be statistically mutually independent. Then $\phi(\underline{X})$ is the random variable representing the system state of the MCS having structure function ϕ, with

(4.2)
$$P[\phi(\underline{X}) = j] = p_j, \ j = 0, 1, \ldots, M,$$
$$P[\phi(\underline{X}) \leq j] = P(j), \ j = 0, 1, \ldots, M.$$

P represents the performance distribution of the system. Let $h = E\phi(\underline{X})$; we may express h as follows:

$$h \equiv h_{\underline{P}}(P_1, \ldots, P_n),$$

since h is a function of the $P_1, \ldots, P_n$. We may also express h alternatively:

$$h \equiv h_{\underline{p}}(\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_n),$$

where $p_i \equiv (p_{io}, p_{i1}, \ldots, p_{iM})$ for $i = 1, \ldots, n$. In either case we call h the performance function of the system.

The following identity expresses a system performance function of n components in terms of system performance functions of n-1 components.

Lemma 4.1. The following identity holds for h:

(4.3) $h(\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_n) = \sum\limits_{j=0}^{M} p_{ij} h(j_i; \underline{p}_1, \ldots, \underline{p}_n), \quad i=1, \ldots, n,$

where $h(j_i; \underline{p}_1, \underline{p}_2, \ldots, \underline{p}_n) = E\phi(j_i, \underline{X}) \equiv E\phi(X_1, \ldots, X_{i-1}, j, X_{i-1}, \ldots, X_n).$

<u>Proof</u>. By Lemma 3.1 and the mutual independence of the components, we have:

$$E\phi(\underline{X}) = \sum\limits_{j=0}^{M} EI_{[X_i = j]} \cdot E\phi(j_i, \underline{X}).$$

Relation (4.3) follows immediately. $\|$

The following theorem shows that h is strictly increasing in each p_{ij} for $j > 0$.

<u>Theorem 4.1</u>. Let $h(\underline{p}_1, \ldots, \underline{p}_n)$ be the performance function of an MCS. Let $0 < p_{ij} < 1$ for $i=1, \ldots, n$ and $j=0, 1, \ldots, M$. Then $h(\underline{p}_1, \ldots, \underline{p}_n)$ is strictly increasing in p_{ij}, $i=1, \ldots, n$ and $j=1, \ldots, M$.

<u>Proof</u>. From (4.3) and the fact that $\sum\limits_{j=0}^{M} p_{ij} = 1$, $i=1, \ldots, n$, we have

$$h(\underline{p}_1, \ldots, \underline{p}_n) = \sum\limits_{j=0}^{M} p_{ij} E[\phi(j_i, \underline{X}) - \phi(0_i, \underline{X})].$$

Thus

$$\frac{\partial h}{\partial p_{ij}} = E[\phi(j_i, \underline{X}) - \phi(0_i, \underline{X})], \quad i=1, \ldots, n \text{ and } j=1, \ldots, M.$$

Since ϕ is increasing, $E[\phi(j_i, \underline{X}) - \phi(0_i, \underline{X})] \geq 0$. In addition, $\phi(j_i, \underline{x}^\circ) - \phi(0_i, \underline{x}^\circ) > 0$ for some $(\cdot_i, \underline{x}^\circ)$ since the structure is an MCS. Since $0 < p_{ij} < 1$ for all i and j, we have $E[\phi(j_i, \underline{X}) - \phi(0_i, \underline{X})] > 0$. Thus the desired result follows. $\|$

Next we obtain properties of h as a function of the $P_1, \ldots, P_n$. First we show that $h(P_1, \ldots, P_n)$ is monotone increasing with respect to stochastic ordering. A similar result is proved by Barlow [2] for his subclass of

MCS's (see Ex. 3.4). Our proof of the more general result is simpler.

__Theorem 4.2.__ Let P_i, P_i' be two possible performance distributions for component i, i=1,...,n. Assume $P_i(j) \geq P_i'(j)$ for j=0,1,...,M and i=1,...,n. Let P (P') be the corresponding system performance distribution. Then

(i) $P(j) \geq P'(j)$ for j=0,1,...,M,

(ii) $h(P_1,...,P_n) \leq h(P_1',...,P_n')$.

__Proof.__ Let $X_1,...,X_n (X_1',...,X_n')$ be mutually independent random variables having distribution functions $P_1,...,P_n (P_1',...,P_n')$ respectively. Then for i=1,...,n, $P_i(j) \geq P_i'(j)$ for j=0,1,...,M implies that $X_i \overset{st}{<} X_i'$. Since ϕ is increasing, $\phi(\underline{X}) \overset{st}{<} \phi(\underline{X}')$. The desired results in (i) and (ii) follow immediately. $\parallel$

Similarly, we relate properties of P, the system performance distribution, to properties of h, the system performance function, or to properties of the p_{ij}. As examples we state the following two straightforward results.

__Lemma 4.2.__ Let h be the performance function and P be the performance distribution of an MCS. Then

$$h = \sum_{j=0}^{M-1} \bar{P}(j), \quad \text{where } \bar{P}(j) \equiv 1 - P(j).$$

__Proof.__ Since $\phi(\underline{X})$ is a nonnegative integer valued random variable, then $E\phi(\underline{X}) = \sum_{j=0}^{\infty} P[\phi(\underline{X}) > j]$, yielding the desired result. $\parallel$

A decomposition identity is given in:

__Theorem 4.3.__ Let ϕ be the structure function of a MCS. Then

$$(4.4) \qquad P[\phi(\underline{X}) > \ell] = \sum_{j=0}^{M} p_{ij} P[\phi(j_i, \underline{X}) \geq \ell], \quad \ell=1,...,M.$$

__Proof.__ By the law of total probability, we have:

$$P[\phi(\underline{X}) \geq \ell] = \sum_{j=0}^{M} P[\phi(\underline{X}) \geq \ell | X_i = j] \, P[X_i = j].$$

Since the components are mutually independent, (4.4) follows immediately. ‖

Next we obtain bounds on both the system performance function and the system performance distribution. Using Theorem 3.1, we establish:

<u>Theorem 4.4</u>. Let P be the performance distribution and h be the performance function of an MCS. Let P_i be the ith component performance distribution for i=1,...,n. Then for j=0,1,...,M-1:

$$\text{(i).} \quad \prod_{i=1}^{n} P_i(j) \leq P(j) \leq 1 - \prod_{i=1}^{n} \bar{P}_i(j),$$

$$\text{(ii).} \quad \sum_{j=1}^{M} \prod_{i=1}^{n} \bar{P}_i(j-1) \leq h \leq \sum_{j=1}^{M} [1 - \prod_{i=1}^{n} P_i(j-1)].$$

<u>Proof</u>. (i). By Theorem 3.1 we have $\min_{1 \leq i \leq n} X_i \leq \phi(\underline{X}) \leq \max_{1 \leq i \leq n} X_i$. Since $X_1,...,X_n$ are mutually independent, (i) follows immediately.

(ii). We use the fact that $h = \sum_{j=1}^{M} P[\phi(\underline{X}) \geq j]$ and the bounds on $P[\phi(\underline{X}) \geq j]$. ‖

Next we illustrate how we can use the upper cticial connection vectors to establish bounds on the system performance distribution P and consequently on system performance function h. Let $\underline{y}_1^j,...,\underline{y}_{n_j}^j$ be the upper critical connections to level j, j=0,1,...,M. Let A_r^j denote the event that $\underline{X} \geq \underline{y}_r^j$, r=1,...,$n_j$. By Theorem 3.3, it follows that $P[\phi(\underline{X}) \geq j] = P[\bigcup_{r=1}^{n_j} A_r^j]$.

Let $S_k = \sum_{1 \leq i_1 < i_2 < ... < i_k \leq n_j} P[A_{i_1}, A_{i_2} ... A_{i_k}]$. By the inclusion-exclusion principle,

$$P[\phi(\underline{X}) \geq j] = \sum_{k=1}^{n_j} (-1)^{k-1} S_k.$$

Thus

$$P[\phi(\underline{X}) \geq j] \leq S_1 = \sum_{r=1}^{n_j} P[A_r],$$

$$P[\phi(\underline{X}) \geq j] \geq S_1 - S_2,$$

and so on, constituting upper and lower bounds on $P[\phi(\underline{X}) \geq j] = \bar{P}(j-1)$ for $j=1,\ldots,n$. Since $h = \sum_{j=1}^{M} P[\phi(\underline{X}) \geq j]$; we antomatically have upper and lower bounds on h also. Note that $P(A_r^j) = P[\underline{X} \geq \underline{y}_r^j] = P[X_1 \geq y_{1r}^j, X_2 \geq y_{2r}^j, \ldots, X_n \geq y_{nr}^j]$ $= \prod_{i=1}^{n} P[X_i \geq y_{ir}^j]$ for $1 \leq r \leq n_j$ and $j=1,\ldots,M$.

5. Dynamic Models for Multistate Coherent Systems.

In the last two sections, we have studied deterministic and probabilistic properties of MCS's at a fixed point in time. Now we consider dynamic models, i.e., models in which the state of the system and of its components vary over time. At time 0, the system and each of its components are in state M (perfect functioning). As time passes, the performance of each component (and consequently of the system itself) deteriorates to successively lower levels, until ultimately failure occurs (level 0 is attained).

In the binary case, the length of time during which a component or system functions is called the lifelength of the component or system; these lifelengths are nonnegative random variables. The corresponding lifelength distributions have been classified according to various notions, of aging. See, e.g., [1]. Two of the important classes of life distributions are the increasing failure rate average (IFRA) class and the new better than used (NBU) class. Closure of these classes under various basic reliability operations, such as convolution of distributions and formation of binary coherent systems, are investigated in [1]. In this section we investigate generalizations of these useful concepts in the multistate case.

Let $\{X_i(t), t \geq 0\}$ denote the stochastic process representing component i state at time t as t varies over the non-negative real numbers for $i=1,\ldots,n$. The stochastic process $\{\phi(\underline{X}(t)), t \geq 0\}$ represents the corresponding

system state as t varies over the non-negative real numbers, where $\underline{X}(t)$ = $(X_1(t),\ldots,X_n(t))$. We assume $X_i(0) = M$, $i=1,\ldots,n$. We assume as before that components are mutually statistically independent; thus the process $\{X_i(t),\ t\geq 0\}$, $i=1,\ldots,n$, are also mutually independent.

Following Ross [12], we present:

<u>Definition 5.1</u>. The stochastic process $\{X_i(t),\ t\geq 0\}$ is said to be <u>IFRA process</u> if $T_i^j \equiv \inf\{t\colon X_i(t)\leq j\}$ is an IFRA random variable for $j=0,1,\ldots,M-1$.

In a similar fashion we may define an IFRA process for the system: $\{\phi(\underline{X}(t)),\ t\geq 0\}$. Note that in the binary case, T_i^0 is simply the lifelength of component i.

The following theorem is due to Ross [12].

<u>Theorem 5.1</u>. The Generalized IFRA Closure Theorem.

Let $\{X_i(t),\ t\geq 0\}$, $i=1,\ldots,n$, be independent IFRA processes and ϕ an increasing structure function. Then $\{\phi(\underline{X}(t)),\ t\geq 0\}$ is an IFRA process.

We now give a definition for NBU stochastic processes different from the one given by Ross [12]. We then derive a simple characterization for our NBU stochastic processes. Using this characterization, we give a simple proof of a generalized NBU closure theorem.

<u>Definition 5.2</u>. The stochastic process $\{X_i(t),\ t\geq 0\}$ is an <u>NBU stochastic process</u> if T_i^j is an NBU random variable for $j=0,1,\ldots,M-1$.

In a similar fashion we may define an NBU stochastic process $\{\phi(\underline{X}(t)),\ t\geq 0\}$ for the system.

The following lemma gives a simple characterization for an NBU process (as well as for an NBU random variable). We omit the simple proof.

<u>Lemma 5.1</u>. The stochastic process $\{X(t),\ t\geq 0\}$ is NBU if and only if

for all $s \geq 0$ and $t \geq 0$:

$$X(s+t) \overset{st}{<} \min(X'(s), X'(t)),$$

where $X'(s)$ and $X'(t)$ are two independent random variables having the same distributions as $X(s)$ and $X(t)$ respectively.

We may now prove the main result of this section.

<u>Theorem 5.2</u>. Let ϕ be the structure function of an MCS having n components. Let $\{X_i(t), t \geq 0\}$, $i=1,\ldots,n$ be independent NBU stochastic processes. Then $\{\phi(\underline{X}(t), t \geq 0\}$ is an NBU stochastic process.

<u>Proof</u>. For arbitrary fixed $s \geq 0$ and $t \geq 0$, let $X_1'(s),\ldots,X_n'(s)$, $X_1'(t),\ldots,X_n'(t)$ be mutually independent random variables having the same distributions as $X_1(s),\ldots,X_n(s)$, $X_1(t),\ldots,X_n(t)$ respectively. Since $\{X_i(t), t \geq 0\}$ is an NBU process, we have by Lemma 5.1:

$$X_i(s+t) \overset{st}{<} \min(X_i'(s), X_i'(t)), \quad i=1,\ldots,n.$$

Since ϕ is increasing, it follows that

$$\phi(\underline{X}(s+t)) \overset{st}{<} \phi(\min(\underline{X}'(s),\underline{X}'(t)).$$

By Theorem 3.2, $\phi(\min(\underline{X}'(s),\underline{X}'(t)) < \min(\phi(\underline{X}'(s)), \phi(\underline{X}'(t)))$. Thus $\phi(\underline{X}(s+t)) \overset{st}{<} \min(\phi(\underline{X}'(s)), \phi(\underline{X}'(t)))$. Using Lemma 5.1 again, the desired result follows. $\|$

<u>Remark 5.1</u>. The useful characterization of Lemma 5.1 makes our proof of the generalized NBU closure theorem in the binary case simpler than the proof given in [1].

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The vast majority of reliability analyses assume that components and system are in either of two states: functioning or failed. The present paper develops basic theory for the study of systems of components in which any of a finite number of states may occur, representing at one extreme perfect functioning and at the other extreme complete failure. We lay down axioms extending the standard notion of a coherent system to the new notion of a multistate coherent system. For such systems we obtain deterministic and

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probabilistic properties for system performance which are analogous to well-known results for coherent system reliability.